Extreme statistics of non-intersecting Brownian motions and random matrix theory

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- S. N. Majumdar (LPTMS, Orsay)
- J. Rambeau (LPT, Orsay)
- J. Randon-Furling (Univ. Paris 1, Paris)

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References:

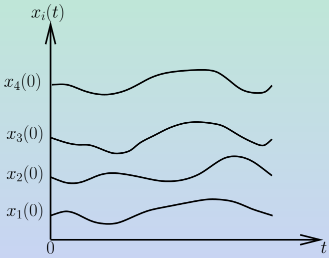
- G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling, Phys. Rev. Lett. **101**, 150601 (2008)
- J. Rambeau, G. S., Europhys. Lett. **91**, 60006 (2010)
- P. J. Forrester, S. N. Majumdar, G. S., Nucl. Phys. B **844**, 500 (2011)
- J. Rambeau, G. S., Phys. Rev. E **83**, 061146 (2011)

- N Brownian motions in one-dimension

$$\dot{x}_i(t) = \zeta_i(t) \ , \ \langle \zeta_i(t) \zeta_j(t') \rangle = \delta_{i,j} \delta(t - t')$$
$$x_1(0) < x_2(0) < \ldots < x_N(0)$$

- Non-intersecting condition

$$x_1(t) < x_2(t) < \ldots < x_N(t) \ , \quad \forall t \geq 0$$



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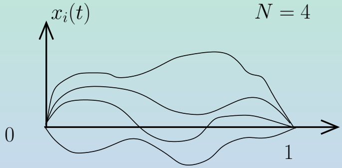

watermelons

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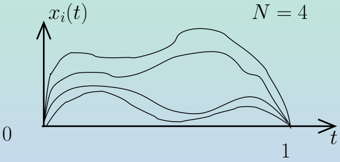


watermelons "with a wall"

Vicious walkers in physics

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 48, NUMBER 5 1 MARCH 1968

Soluble Model for Fibrous Structures with Steric Constraints

P.-G. de Gennes

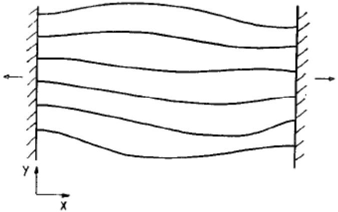
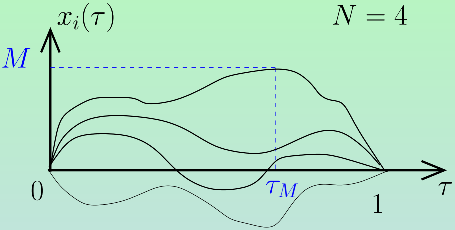


Fig. 1. Model for a two-dimensional fiber structure. The component chains are assumed to be attached to two plates I and F and placed under tension. The chains are bent by thermal fluctuations. Different chains cannot intersect each other.

Vicious walkers in physics

- P. G. de Gennes, *Soluble Models for fibrous structures with steric constraints* (1968)
- M. E. Fisher, *Walks, Walls, Wetting and Melting* (1984)
- D. J. Grabiner, *Brownian motion in a Weyl chamber, non-colliding particles, and random matrices* (1997)
- C. Krattenthaler, *Asymptotics for random walks in alcoves of affine Weyl groups* (2003)
- H. Spohn, M. Praehofer, P. L. Ferrari et al. *Stochastic growth models* (2006)
- ...

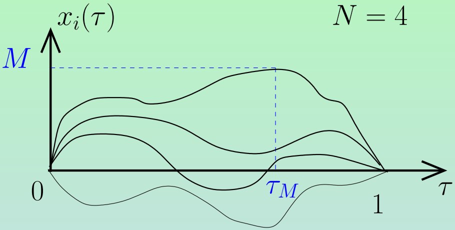
Extreme statistics of vicious walkers



$$x_1(t) < x_2(t) < ... < x_N(t)$$
$$M = \max_{\tau}[x_N(\tau), 0 \le \tau \le 1]$$
$$x_N(\tau_M) = M$$

$P_N(M, \tau_M) \equiv$ joint probability distribution function of M, τ_M

Extreme statistics of vicious walkers



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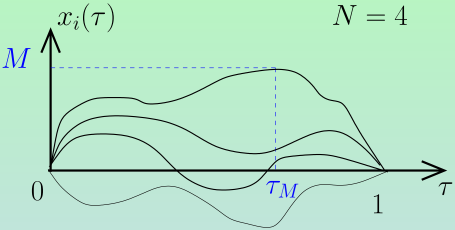

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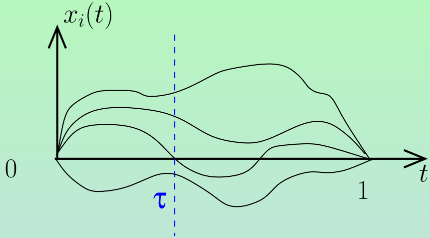
Q1 : Can one compute $P_N(M, \tau_M)$?

Q2 : Asymptotics of $P_N(M, \tau_M)$ for large N ?

Outline

Outline

Non intersecting Brownian motions and RMT



- Joint probability of $x_1(\tau), x_2(\tau), \cdots, x_N(\tau)$ at fixed time τ

$$P_{\text{joint}}(x_1, x_2, \cdots, x_N, \tau) \propto \sigma(\tau)^{-N^2} \prod_{i<j=1}^{N} (x_i - x_j)^2 e^{-\frac{1}{\sigma^2(\tau)} \sum_{i=1}^{N} x_i^2}$$

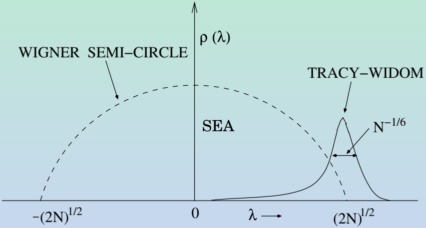
$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of the Gaussian Unitary Ensemble (GUE, $\beta = 2$)

Non intersecting Brownian motions and RMT

- The rescaled positions $\frac{x_i}{\sigma(\tau)}$ are distributed like the eigenvalues of random matrices of Gaussian Unitary Ensemble (GUE, $\beta = 2$)
- Mean density $\rho(\lambda)$ of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_N$ for GUE

$$\rho(\lambda) = \frac{1}{N} \sum_{\alpha=1}^{N} \langle \delta(\lambda - \lambda_\alpha) \rangle$$



Watermelons in the limit of large N

- Consequences for watermelons without wall for large N

$$\frac{x_N(\tau)}{\sqrt{2\tau(1-\tau)}} \sim \sqrt{2N} + \frac{N^{-1/6}}{\sqrt{2}}\chi_2$$

$$\mathrm{Proba}[\chi_2 \leq \xi] = \mathcal{F}_2(\xi),$$ Tracy-Widom distribution for $\beta = 2$

$$\mathcal{F}_2(\xi) = \exp\left(-\int_\xi^\infty (x-\xi)\, q^2(x)\, \mathrm{d}x\right)$$

where $q(x)$ is the Hasting-Mc Leod's solution of the Painlevé II equation

$$q''(x) = x\, q(x) + q^3(x)\ ,\ q(x) \sim \mathrm{Ai}(x)\ ,\ x \to \infty$$

C. A. Tracy, H. Widom '94

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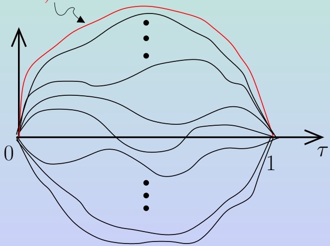
- When $N \to \infty$, $x_N(\tau)$ reaches a circular shape

$x_N(\tau) \sim 2\sqrt{N}\sqrt{\tau(1-\tau)}$

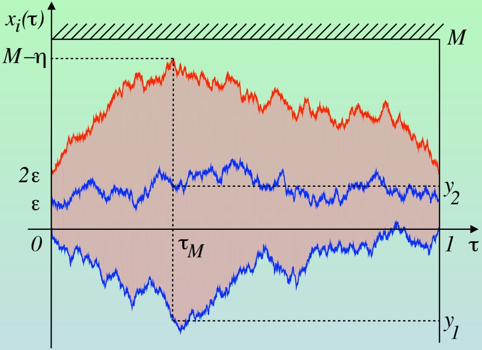


Fluctuations

$$x_N(\tau = 1/2) - \sqrt{N} \sim N^{-\frac{1}{6}}$$

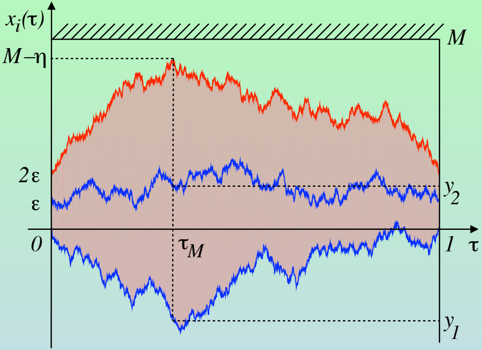
Outline

Feynman-Kac formula



- $$P_N(M,\tau_M) = \lim_{\epsilon,\eta\to 0} \frac{1}{Z_N} \int_{-\infty}^{M-\eta} \mathrm{d}\mathbf{y}\, p_{<M}(\epsilon,1|\mathbf{y},\tau_M) p_{<M}(\mathbf{y},\tau_M|\epsilon,0)\delta(y_N-(M-\eta))$$

Feynman-Kac formula



- $P_N(M, \tau_M) = \lim_{\epsilon,\eta\to 0} \frac{1}{Z_N} \int_{-\infty}^{M-\eta} \mathrm{d}\mathbf{y}\, p_{<M}(\epsilon, 1|\mathbf{y}, \tau_M) p_{<M}(\mathbf{y}, \tau_M|\epsilon, 0)\delta(y_N - (M-\eta))$
- $p_{<M}(\cdot,\cdot|\cdot,\cdot)$: computed using Feynman – Kac

G. S., S.N. Majumdar, A. Comtet, J. Randon-Furling '08

Exact results for N vicious walkers

- Joint distribution of M and τ_M J. Rambeau, G.S, EPL '10, PRE '11

$$P_N(M,\tau_M) = B_N\,[\det \mathbf{D}]\; {}^t\mathbf{U}(\tau_M)\,\mathbf{D}^{-1}\,\mathbf{U}(1-\tau_M)$$

$$\mathbf{D}_{i,j} = (-1)^{i-1} H_{i+j-2}(0) - e^{-2M^2} H_{i+j-2}(\sqrt{2}M)$$

$$\mathbf{U}_i(\tau_M) = \tau_M{}^{-\frac{i+1}{2}} H_i\left(M/\sqrt{2\tau_M}\right)\; e^{-\frac{M^2}{2\tau_M}}$$

$H_i(\cdot) \equiv$ Hermite polynomials

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- Marginal distribution of τ_M

$$N = 2 \;:\quad P_2(\tau_M) = 4\left(1 - \frac{1 + 10\tau_M(1-\tau_M)}{(1 + 4\tau_M(1-\tau_M))^{5/2}}\right)$$

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- Asymptotics for large N ?

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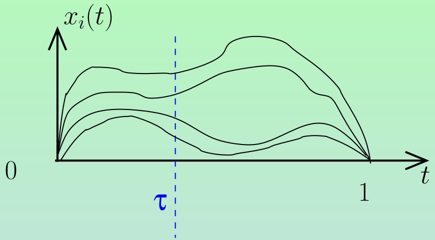
- Asymptotics for large N ? still difficut BUT ...

Outline

Watermelons with a wall and RMT



- At fixed time τ

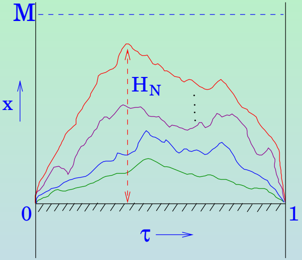
$$P_{\rm joint}(\mathbf{x},\tau) \propto \sigma(\tau)^{-N(2N+1)} \prod_{i=1}^{N} x_i^2 \prod_{1\le i<j\le N} (x_i^2 - x_j^2)^2 e^{-\frac{\mathbf{x}^2}{\sigma^2(\tau)}}$$

$$\sigma(\tau) = \sqrt{2\tau(1-\tau)}$$

- The rescaled positions $y_i = \frac{x_i^2}{2\sigma^2(\tau)}$ are distributed like the eigenvalues of Wishart matrices, with $\beta = 2$ and $M - N = \frac{1}{2}$

- Cumulative distribution of the maximal height

$$
\begin{aligned}
F_N(M) &= \Pr\left[x_N(\tau) \le M \,,\ \forall\, 0 \le \tau \le 1\right] \\
&= \int_0^1 d\tau_M \int_0^M dx\, P_N(x, \tau_M)
\end{aligned}
$$



Maximal height of watermelons with a wall

- Cumulative distribution of the maximal height

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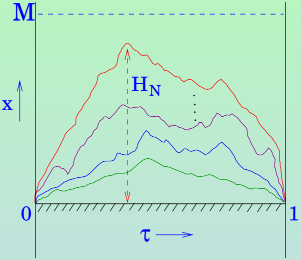

- Feynman-Kac formula

G. S., S. N. Majumdar, A. Comtet, J. Randon-Furling '08

$$
F_N(M) = \frac{A_N}{M^{2N^2+N}} \sum_{n_1,\cdots,n_N=0}^{+\infty} \prod_{i=1}^{N} n_i^2 \prod_{1\leq j<k\leq N} (n_j^2 - n_k^2)^2 e^{-\frac{\pi^2}{2M^2}\sum_{i=1}^N n_i^2}
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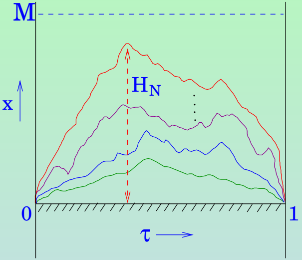

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What about the asymptotic behavior of $F_N(M)$ for $N \to \infty$?

Large N asymptotics for $F_N(M)$

- Using discrete orthogonal polynomials (Gross-Matytsin '94, Crescimanno-Naculich-Schnitzer '96) one shows

$$\frac{d^2}{dt^2} \log F_N\left(\sqrt{2N}(1 + t/(2^{7/3} N^{2/3}))\right) = -\frac{1}{2}\left(q^2(t) + q'(t)\right)$$

$$q''(t) = 2q^3(t) + t\, q(t)\ ,\ \ q(t) \sim \mathrm{Ai}(t)\ ,\ \ t \to \infty$$

i.e.

$$F_N(M) \quad \to \quad \mathcal{F}_1\left(2^{11/6} N^{1/6} \left|M - \sqrt{2N}\right|\right)$$

$$\mathcal{F}_1(t) \quad = \quad \exp\left(-\frac{1}{2}\int_t^\infty \left((s-t)\, q^2(s) - q(s)\right)\, ds\right)$$

$$\equiv \quad \text{Tracy-Widom distribution for } \beta = 1$$

P. J. Forrester, S. N. Majumdar, G.S. '11

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P. J. Forrester, S. N. Majumdar, G.S. '11

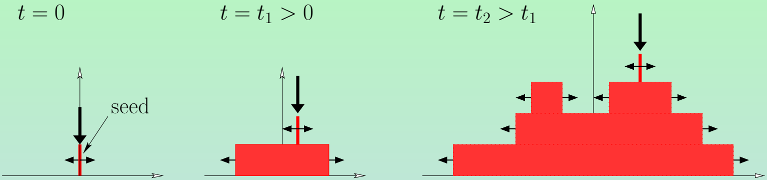
- Also interesting results for large deviations

P. J. Forrester, S. N. Majumdar, G.S. '11

Outline

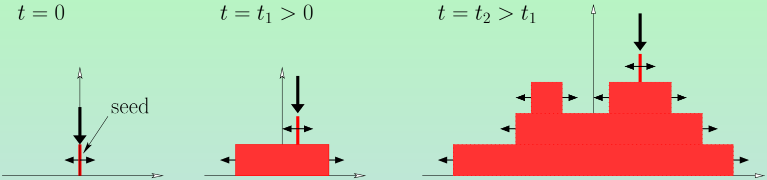
Curved growing interface : the PNG droplet

- Polynuclear Growth Model

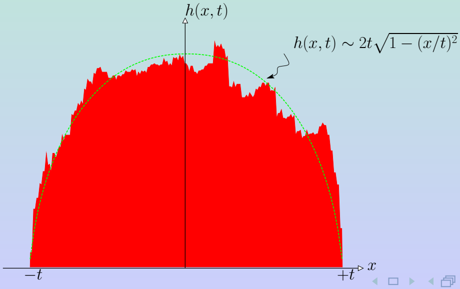


Curved growing interface : the PNG droplet

- Polynuclear Growth Model

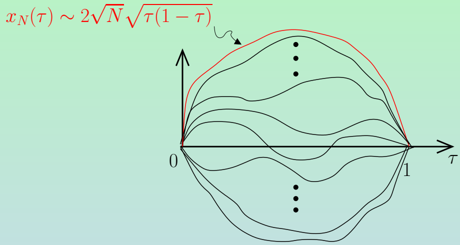


- At large time t the profile becomes droplet-like

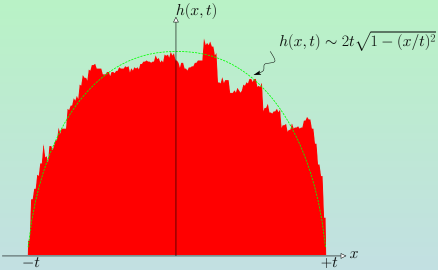


Vicious walkers and PNG droplet

watermelons



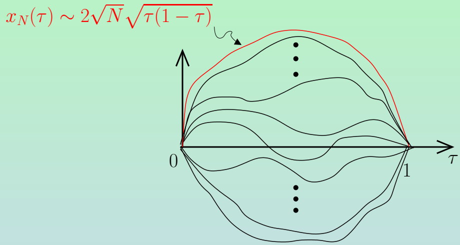
PNG droplet

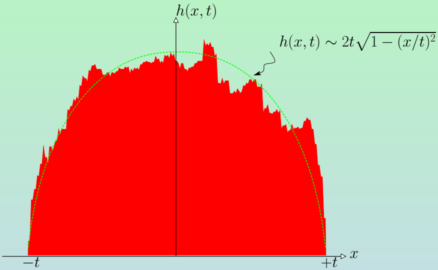


$$x_N \iff h$$
$$\tau \iff x$$
$$N \iff t$$

watermelons

PNG droplet





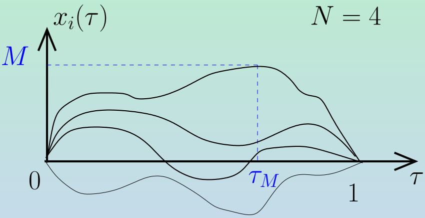
$$\frac{h(ut^{\frac{2}{3}},t)-2t}{t^{\frac{1}{3}}} \stackrel{d}{=} \frac{x_N(\frac{1}{2}+\frac{u}{2}N^{-\frac{1}{3}})-\sqrt{N}}{N^{-\frac{1}{6}}} \stackrel{d}{=} \mathcal{A}_2(u)-u^2$$

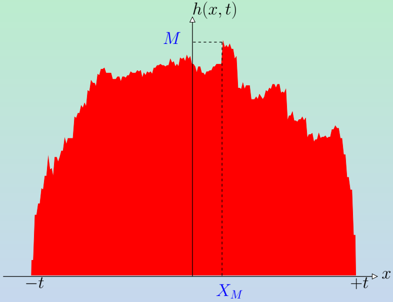
Prähofer & Spohn '00

$\mathcal{A}_2(u) \equiv$ Airy$_2$ process

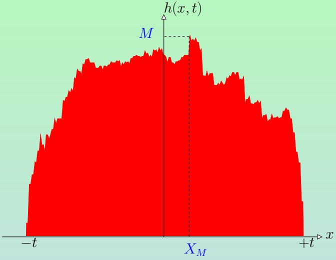
Vicious walkers and PNG droplet

- Use this correspondence to study extreme statistics of PNG





Consequences for curved stochastic growth

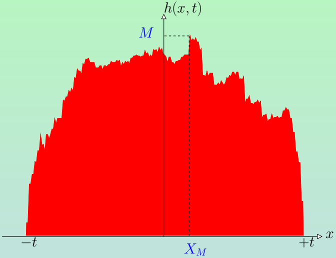


- Distribution of the height field $h(0,t)$ Prähofer & Spohn '00

$$\lim_{t\to\infty} P\left(\frac{h(0,t)-2t}{t^{1/3}} \leq s\right) = \mathcal{F}_2(s)$$

$\mathcal{F}_2(s) \equiv$ Tracy – Widom distribution for $\beta = 2$

Consequences for curved stochastic growth



- Maximum $M \equiv \max_{-t \leq x \leq t} h(x,t)$ P. Forrester, S. N. Majumdar, G. S. NPB '11

$$\lim_{t\to\infty} P\left(\frac{M - 2t}{t^{1/3}} \leq s\right) = \mathcal{F}_1(s)$$

$\mathcal{F}_1(s) \equiv$ Tracy – Widom distribution for $\beta = 1$

Consequences for curved stochastic growth

- Maximum $M \equiv \max_{-t \leq x \leq t} h(x,t)$ P. Forrester, S. N. Majumdar, G. S. NPB '11

$$\lim_{t\to\infty} P\left(\frac{M-2t}{t^{1/3}} \leq s\right) = \mathcal{F}_1(s)$$

$$\mathcal{F}_1(s) \equiv \text{Tracy} - \text{Widom distribution for } \beta = 1$$

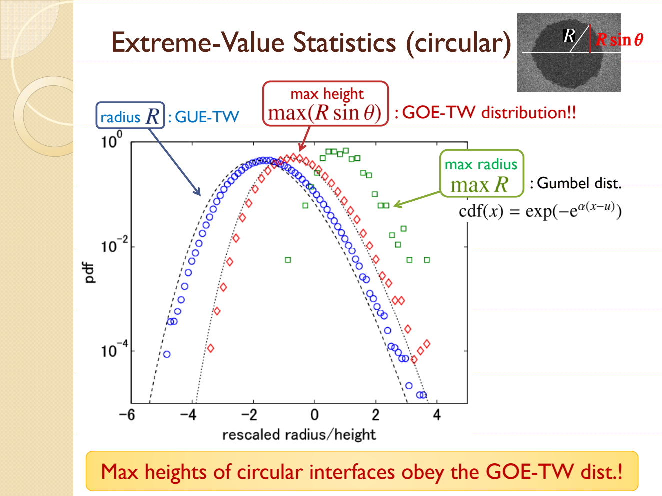
see also

  - Krug *et al.* '92, Johansson '03 (indirect proof),
  - G. M. Flores, J. Quastel, D. Remenik, arXiv:1106.2716

Experiments on nematic liquid crystals

K. A. Takeuchi, M. Sano, Phys. Rev. Lett. **104**, 230601 (2010)

K. A. Takeuchi *et al.*, Sci. Rep. **1**, 34 (2011)



Outline

Conclusion

- Connection between vicious walkers and random matrices
- The maximal height is given, for $N \to \infty$, by the Tracy-Widom distribution $\beta = 1$
- It can be measured experimentally through the connection with stochastic growth models

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- It can be measured experimentally through the connection with stochastic growth models
- What about the distribution of the position of the maximum for large N ?

  see recent results by G. M. Flores, J. Quastel, D. Remenik, arXiv:1106.2716